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A simple proof of a recurrence theorem for random walks in \mathbb{Z}^2

Jean-Marc Derrien

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Abstract *In this note, we prove without using Fourier analysis that the symmetric square integrable random walks in \mathbb{Z}^2 are recurrent.*

George Pólya related in [3, pp. 582-583] an incident that enables him to formulate the question of recurrence for random walks : during a stroll through the woods, he felt embarrassed because he met “certainly much too often” a student with his girlfriend.

Let x be an element of \mathbb{Z}^2 . A random walk in \mathbb{Z}^2 starting at x is a sequence $(S_n)_{n \in \mathbb{N}}$ of random variables such that

$$S_0 = x \quad \text{and} \quad S_n = x + X_1 + X_2 + \cdots + X_n, \quad n \in \mathbb{N},$$

where $(X_n)_{n \in \mathbb{N}}$ is a sequence of independent and identically distributed (i.i.d.), \mathbb{Z}^2 -valued random variables. $(S_n)_{n \in \mathbb{N}}$ is characterized in law by x and the law of X_1 .

In this note, we give an elementary proof of the following result.

Theorem *If $(S_n^{(1)})_{n \in \mathbb{N}}$ and $(S_n^{(2)})_{n \in \mathbb{N}}$ are two square integrable, independent and identically distributed random walks in \mathbb{Z}^2 , then*

$$S_n^{(1)} = S_n^{(2)} \quad \text{infinitely often}$$

with probability one.

This theorem is a straightforward consequence of the following proposition (first proved in [2] in a more general context) applied to $(S_n := S_n^{(1)} - S_n^{(2)})_{n \in \mathbb{N}}$.

Proposition *If $(S_n)_{n \in \mathbb{N}}$ is a symmetric, square integrable random walk starting at 0 in \mathbb{Z}^2 , then*

$$S_n = 0 \quad \text{infinitely often}$$

with probability one.

Proof It is classical that, in order to prove this proposition, we have only to establish that

$$\sum_{n=0}^{+\infty} \mathbb{P}[S_n = 0] = +\infty$$

(see, for instance, [1]).

One can write

$$S_n = X_1 + X_2 + \cdots + X_n,$$

where $(X_k)_{k \geq 1}$ is a sequence of i.i.d., square integrable, \mathbb{Z}^2 -valued random variables with $X_1 \equiv -X_1$ in distribution.

Since the square integrable random walk $(S_n)_{n \in \mathbb{N}}$ is symmetric, it is centered and we have

$$\mathbb{E}(\|S_n\|_2^2) = \sum_{k=1}^n \mathbb{E}(\|X_k\|_2^2) + \sum_{1 \leq k < l \leq n} \mathbb{E}(X_k) \cdot \mathbb{E}(X_l) = n \mathbb{E}(\|X_1\|_2^2)$$

($\|\cdot\|_2$ denotes the euclidean norm).

The symmetry also gives

$$\begin{aligned} \mathbb{P}[S_{2n} = 0] &= \sum_{x \in \mathbb{Z}^2} \mathbb{P}[S_{2n} = 0 \mid S_n = x] \mathbb{P}[S_n = x] \\ &= \sum_{x \in \mathbb{Z}^2} \mathbb{P}[-X_{n+1} - X_{n+2} - \dots - X_{2n} = x] \mathbb{P}[S_n = x] = \sum_{x \in \mathbb{Z}^2} \mathbb{P}[S_n = x]^2. \end{aligned}$$

Hence if we introduce

$$B_n := \{x \in \mathbb{Z}^2 : \|x\|_2^2 < 2n \mathbb{E}(\|X_1\|_2^2)\},$$

we deduce from Cauchy-Schwarz's and Markov's inequalities that, if n is large enough,

$$\begin{aligned} \mathbb{P}[S_{2n} = 0] &\geq \frac{1}{|B_n|} \left(\sum_{x \in B_n} \mathbb{P}[S_n = x] \right)^2 \\ &\geq \frac{C}{n} \left(1 - \mathbb{P}[\|S_n\|_2^2 \geq 2n \mathbb{E}(\|X_1\|_2^2)] \right)^2 \\ &\geq \frac{C}{n} \left(1 - \frac{\mathbb{E}(\|S_n\|_2^2)}{2n \mathbb{E}(\|X_1\|_2^2)} \right)^2 = \frac{C}{4n}, \end{aligned}$$

where $C > 0$ depends only on $\mathbb{E}(\|X_1\|_2^2)$. The proposition follows. •

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